Calculus: What is it? What is it for?

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The tone of this essay is conversational. The best way to learn from it is to read it slowly and carefully, section by section. Keep a list of questions. You are welcome to contact the authors for clarification.

The reader is assumed to have no knowledge of calculus, but to understand the basic algebra of functions, particularly linear functions. Sources for this information include:

- any text or review book covering College Algebra
A. SLOPE FINDING

Differential Calculus, despite its complex sounding name, is really nothing more than "slope finding." You already know a good bit about slope finding (See “Quantitative Skills Interactive” CD-ROM).

You already understand that we find the slope of a line by using the formula

\[
\text{Slope formula for a line } \quad \frac{y_2 - y_1}{x_2 - x_1}
\]

We need to emphasize that two points are needed to find the slope of a line but that any two points will give the same slope number. This is what we mean when we say that a line has constant slope.

Calculating slope using points P(1, 4) and Q(2, 6):

\[
\text{Slope using points P}(1, 4) \text{ and Q}(2, 6)
\]

\[
is \quad \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 4}{2 - 1} = \frac{2}{1} = 2
\]

\[
\text{Calculating slope using points P and R:}
\]

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{1 - (-1)} = \frac{4}{2} = 2
\]
Slope tells about the steepness of lines and about their directions.

Notice that Line #1 (slope is 2) is steeper than Line #2 (slope is $\frac{1}{2}$).

Notice that Line #3 slopes downward and has a slope of $-1$. 
B. SLOPE AND RATE

Another way of writing the formula for slope is

\[
\text{Slope} = \frac{\Delta y}{\Delta x}
\]

Where "\(\Delta y\)" ("delta y") means change in the y-direction (or, in the vertical direction) and "\(\Delta x\)" ("delta x") means change in the x-direction (or, in the horizontal direction).

As we move along the line from point \(P_1\) to point \(P_2\), we have gone \(\Delta y\) units up, and \(\Delta x\) units to the right. So the slope of the line can be calculated by \(\frac{\Delta y}{\Delta x}\).

![Diagram showing slope and rate calculations](image)

Notice that \(\Delta y\) is a name for the quantity \((y_2 - y_1)\) and, similarly, \(\Delta x\) is a name for the quantity \((x_2 - x_1)\).
**Interesting Question:** What does it mean for a line to have a slope of 2?

**Answer:** Looking at the diagram, you can see that as we move along the line from left to right, a change in the \(x\)-direction of 1 unit is *always* accompanied by a change in the \(y\)-direction of 2 units.

\[
\text{Slope} = 2 \quad \Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{2}{1}
\]

Compare a line with a slope of +2 to one with a slope of -2.

\[
\text{Slope} = -2 \quad \Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{-2}{1}
\]

In this case, an increase of 1 unit in the \(x\)-direction (i.e., \(\Delta x = +1\)) is accompanied by a decrease of 2 units in the \(y\)-direction (i.e., \(\Delta y = -2\)). And therefore slope of the line is \(\frac{\Delta y}{\Delta x} = \frac{-2}{1} = -2\).
Since slope is equal to \( \frac{\text{change in } y}{\text{change in } x} \), the slope is describing

the rate of change in \( y \) with respect to (changes in) \( x \).

Since a straight line has constant slope, we might even choose to define a line as a curve with a constant rate of change. Some texts do.

Read the following three examples carefully. They help to demonstrate that slope can be understood as a rate of change. (You will need to remember that if the equation of a line is given as \( y = mx + b \) then "\( m \)" turns out to be the slope of the line.)
Example 1: **Women in the Labor Force** – In a certain region, the number of women in the labor force is expected to increase during the years 2000-2010. One forecasting consultant uses the linear model

\[ N = 29.6 + 1.20t \]

...to predict the number of women between the ages of 35 and 44 who will be in the labor force. In this equation, \( N \) equals the number of women (ages 35 to 44) in the labor force (measured in thousands) and \( t \) equals time measured in years since 2000 (\( t = 0 \) corresponds to 2000).

The slope of this line is 1.20

\[ \frac{\Delta N}{\Delta t} = 1.20 \]

Thus if \( \Delta t = 1 \), then we will have \( \Delta N = 1.20 \). This means that for each additional year since 2000, we predict that the number of women (ages 35 - 44) in the labor force will increase by 1.20 thousand or 1,200 women.

According to the model, the number of women (ages 35 - 44) in the labor force is increasing at the rate of 1,200 per year.
Some things to think about:

- Sometimes a student will say that the ‘1.20’ means that the number of women in the labor force is increasing at the rate of 20% per year. But for a linear model, y is always changing by a constant amount, not by a constant percent.

- Consider the percent changes from year to year. They aren’t constant. Are those percents going up or going down?

- What would the graph look like IF N were changing by a constant percent?
Example 2: Marginal Cost

A firm’s Total Cost function is given by \( TC = 10q + 500 \)

with \( q \) = # of units of output and \( TC \) = total cost of producing \( q \) units ($)

The slope of 10 implies that with each additional unit of output, Total Cost increases by $10. Or, the Total Cost is increasing **at the rate of $10 per additional unit produced**.

In business and economics language, we say **Marginal Cost is $10** at every level of output.
Example 3  **Straight Line Depreciation**  The value of a police patrol car is expected to decline over time. This relationship is estimated to be

\[ V = -3,500 \, t + 20,000 \]

where \( V \) = value of the car in dollars and \( t \) = age of car in years

A slope of -3,500 implies that with each additional year of ownership the value of the car decreases by $3,500. Or, the value of the car is **decreasing at the rate of $3,500 per year**.

To summarize, the slope of a line is the rate of change of \( y \) (or the variable on the vertical axis) with respect to \( x \) (or the variable on the horizontal axis). For a straight line, slope is constant.

For instance, in Example 1, the number of females in the labor force is increasing at the constant rate of 1,200 per year. In Example 2, Marginal Cost is at the constant rate of $10 per additional item produced. The patrol car of Example 3 is depreciating at the constant rate of $3,500 per year.
C. SLOPE FINDING FOR A CURVE

What does "slope" mean for a curve? We will examine the parabola \( y = x^2 \).

It is a simple curve, but it will illustrate the problem and its solution.

Here are some points on this curve:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = x^2 )</th>
<th>Corresponding Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
<td>(-2, 4)</td>
</tr>
<tr>
<td>-1.5</td>
<td>2.25</td>
<td>(-1.5, 2.25)</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>(-1, 1)</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.25</td>
<td>(-0.5, 0.25)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.25</td>
<td>(0.5, 0.25)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>1.5</td>
<td>2.25</td>
<td>(1.5, 2.25)</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>(2, 4)</td>
</tr>
</tbody>
</table>

Graphical Representation of the curve \( y = x^2 \)
Using any two points we can still calculate the slope of the line (or secant) between those two points. For example, the slope between \( P \) and \( Q \) is

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{2 - 1} = \frac{3}{1} = 3
\]

BUT if we use points \( V \) and \( Q \), we get a different slope:

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5
\]

So "the" slope of a curve" makes no sense. **There isn’t just one slope.** In terms of the two-point definition for slope, a curve has *many* slopes.
A way out of the *too many slopes* dilemma is to acknowledge that a curve does indeed have many slopes. In fact, it has a *slope at every point*!

Let’s make a definition: **The slope of a curve at a point is defined to be the slope of the tangent line to the curve at that point.** (Think of the Tangent line at $P$ as the line that b-a-r-e-l-y touches the curve at $P$.)
Now we have a **one-point** definition of slope. We can see that this definition agrees with our notion of steepness. Notice that the **Tangent at R** is **steeper** than the **Tangent at P** and the **curve** is **steeper** at R than at P.

For a line, it is very easy to calculate the slope **given any two points**. For a curve, we have a definition for slope **at a point**. That definition is not operational yet; this means we do not have a way of **calculating** it. We could, of course, take a ruler, draw a tangent line and measure to get the slope of that tangent. But it is hard to do this accurately.

Therefore the slope of the Tangent at \( P = \frac{1.7}{2.1} = 0.8095 \)

What we would really like is a **FORMULA** so that we could easily calculate the slope at **any point** on the curve. Read on to see how we can get this formula.
D. GETTING THE FORMULA FOR SLOPE AT A POINT

Look at point P(1, 1) on the curve below. We will draw a line (called a secant) through P and point Q(2, 4) as a start. We have already computed the slope of this Secant PQ.

The slope of this Secant PQ is \( \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{2 - 1} = \frac{3}{1} = 3 \). This will be our first secant, Secant #1.

Note that the slope of 3 does not represent slope at P or at Q.

Note also that Secant #1 is steeper than the Tangent at P. So the slope of the Tangent at P has to be less than 3.
Now we will concentrate on point $P$ again, but this time we'll choose our other point closer to $P$ than $Q$. We'll use the point $(1.5, 2.25)$ and calculate the slope of this Secant #2.

\[
\text{Slope of Secant #2} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5
\]

Two Important Ideas:

- Secant #2 is still steeper than the Tangent at $P$. So we know that the slope of the Tangent at $P$ must be less than 2.5
- This Secant #2 is closer to the Tangent at $P$ than was Secant #1 (the one through $Q$).
Suppose we compute the slope of a third secant through $P$ by choosing a point even closer to $P$ than the point where $x = 1.5$. We'll pick the other point to be the point where $x = 1.25$ [and thus $y = (1.25)^2 = 1.5625$].

The slope of Secant #3 is $$\frac{1.5625 - 1}{1.25 - 1} = \frac{0.5625}{0.25} = 2.25$$

Note that 2.25 is still too large for the slope of the Tangent at $P$. But it is a better estimate than the slopes of the Secants #1 and #2.

✔️ To summarize:

1. We want to find the slope of the Tangent at $P$.
2. We can use points on the curve to the right of $P$ to make secant lines through $P$.
3. All of these secant lines will be steeper than the Tangent at $P$.
4. So none of these secant slopes is the exact slope number of the Tangent at $P$.
5. HOWEVER, the closer the other point is to $P$, the better the slope of that secant approximates the slope of the Tangent at $P$. 
E. THE GAME PLAN

Here is the procedure we will follow. We will choose points closer and closer to $P$. Each time we will calculate the slope between that point and $P$ (i.e., the slope of the secant). Since there will always be a point closer to $P$ than the last one we picked, this process can continue forever. The picture looks something like this for the first five secants and Secant #20:

If you take a straight-edge and lay it along each secant in order, you will see that Secant #20 is closest in slope to the slope of the Tangent at $P$. You need to convince yourself that the slope of Secant #21 would be even closer to the slope of the Tangent at $P$.

Next, we examine this unending sequence of secant slopes. Understand that we cannot list them all.
F. THE SLOPE OF THE TANGENT AT P IS 2

Here are the results of calculating some of the secant slopes. Secants #20, #40 and #50 were added to help you see the trend. Do you see a trend?

<table>
<thead>
<tr>
<th>Secant</th>
<th>Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>3.0</td>
</tr>
<tr>
<td>#2</td>
<td>2.5</td>
</tr>
<tr>
<td>#3</td>
<td>2.25</td>
</tr>
<tr>
<td>#4</td>
<td>2.125</td>
</tr>
<tr>
<td>#5</td>
<td>2.0625</td>
</tr>
<tr>
<td>#20</td>
<td>2.00000095 (rounded)</td>
</tr>
<tr>
<td>#40</td>
<td>2.00000000000091 (rounded)</td>
</tr>
<tr>
<td>#50</td>
<td>2.0000000000000018 (rounded)</td>
</tr>
</tbody>
</table>

These secants are getting closer and closer to the tangent at P.
These slope numbers are getting closer and closer to the number 2.

Big Idea:
The secant slopes are getting closer and closer to the number 2, while the secants themselves are getting nearer and nearer to the Tangent at P, so the slope of the Tangent at P must be 2.
(Please read this sentence again.)

WHEW!!
G. SAY IT AGAIN, THIS TIME WITH LIMITS

Observe again. As point \( Q \) approaches \( P \) along the curve -- we write this as \( Q \to P \)

- The secant lines \( PQ \) are getting nearer and nearer to the **Tangent line at \( P \)**. In fact, the Tangent **at \( P \)** is the *limiting position* for these secants.

While at the same time

- The slopes of these secants are getting closer and closer to the number 2.

Both of these observations involve the idea of a **limit of a sequence of "somethings"**. The "somethings" may be numbers, lines, geometric shapes, algebraic expressions or other symbols. Using the limit concept we may say:

- The **limit** of the sequence of **Secant lines \( PQ \)** is the **Tangent line at \( P \)**.

- The **limit** of the sequence of **slopes** of the Secant lines \( PQ \) is the number 2.

We re-write these two ideas as:

- \( \text{LIM (Secants } PQ \text{)} = \text{Tangent at } P \)
  - \( Q \to P \)

- \( \text{LIM (slopes of Secants } PQ\text{)} = 2 \)
  - \( Q \to P \)
H. THE LANGUAGE OF CALCULUS

Using the methods of Sections D-F, we can get the slope at any point on this curve. But this is a very tedious procedure. And remember, we were aiming for a formula, one comparable to the (two-point) formula for the slope of a line.

It turns out that if we focus on a general point on the curve --

Suppose we take \((x, y)\) instead of \(P(1, 1)\)

-- and use a similar limiting process, we will get a formula for the slope of the tangent at any point on the curve.

Here is our graph again, with a very general point \(P(x, y)\). Since \(y = x^2\) describes this parabola, we know that the coordinates of \(P\) are \((x, x^2)\)
Now take a point $Q$ on the parabola that is located $h$ units to the right of $P$.

Notice that the first coordinate of $Q$ is then $(x + h)$.

As an example, if $x = 2$, and $h = 0.25$, then $P$ is the point $(2, 2^2) = (2, 4)$

$Q$ is therefore the point whose first coordinate is $x + h = 2 + 0.25 = 2.25$

So $Q$ is the point $(2.25, 2.25^2) = (2.25, 5.0625)$

If the first coordinate of $Q$ is $(x + h)$, and $Q$ lies on the curve, then the second coordinate of $Q$ is $(x + h)^2$. 
So we have point $P = (x, x^2)$ and point $Q = (x+h, (x+h)^2)$.

Next, we calculate an expression for the slope of Secant $PQ$:

$$\text{Slope of Secant } PQ = \frac{\Delta y}{\Delta x}$$

$$= \frac{(x + h)^2 - x^2}{(x + h) - x}$$

Squaring $(x+h)$;
Doing the subtraction in the denominator

$$= \frac{x^2 + 2xh + h^2 - x^2}{h}$$

The $x^2$ terms drop out

$$= \frac{h(2x + h)}{h}$$

Factoring out $h$

$$= 2x + h$$

Canceling the $h$ in the numerator with the $h$
in the denominator

So the algebraic expression for the slope of Secant $PQ$ is $(2x + h)$.

What we have now is just another slope formula for the secant line connecting any two points on our curve $y = x^2$.

For example, if $P$ is the point $(1, 1)$ and $Q$ is the point $(1.5, 2.25)$

**Using Ordinary Slope Formula:**

Slope of $PQ = \frac{\Delta y}{\Delta x} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$

**Using Slope $= 2x + h$**

$(h$ is the distance that $Q$ is to the right of $P$)

In this case, $x = 1$ and $h = 0.5$, so

Slope of $PQ = 2x + h$

$= 2 \cdot 1 + 0.5 = 2.5$
I. TAKING THE LIMIT

So far, we have just one secant, Secant PQ, and one tangent, the Tangent at P.

What will happen if we keep P fixed, and let Q move along the curve toward P? We can express that motion as Q → P or, Q is approaching P along the curve.

Four Important Consequences of Q → P:

(1) The closer Q is to P, the more alike are their coordinates. So the difference in the x-coordinates and the difference in the y-coordinates are both getting smaller and smaller.

The difference in the y-values, \( \Delta y \), gets smaller and smaller, so closer and closer to zero.

The difference in the x-values, \( \Delta x \), gets smaller and smaller, so closer and closer to zero. We referred to the difference in the x-values as \( h \).
(2) The Secants $PQ$ are rotating clockwise. They look more and more like the \textbf{Tangent at P}. We write:

$$\lim (\text{Secants } PQ) = \text{Tangent at } P$$

(3) A consequence of (2) is

$$\lim \text{(slope formula of Secants } PQ) = \text{slope formula of Tangent at } P$$

You can look back at Section F for the special case that we investigated.
(4) Notice what happens -- algebraically -- to the slope formula for the secants as \( Q \to P \)

The slope formula for secants \( PQ \) is \((2x + h)\)

But as \( Q \to P \), we also see that the difference in the \( x \)-values (or \( h \)) is getting smaller and smaller and….

So in the expression \((2x + h)\), that second term, \( h \), is getting smaller and smaller and smaller…

We write \((2x + h) \to (2x + 0) = 2x\)

Or \(\lim_{Q \to P} (2x + h) = 2x\)

We read this as

"The limit of the expression \((2x + h)\) is the expression \(2x\) as \( Q \) approaches \( P \) along the curve."

Thus we have:

\[
\lim_{Q \to P} (\text{slope formula of Secants } PQ) = \lim_{Q \to P} (2x + h) = 2x
\]
Here’s the **BIG LEAP**!

By combining consequences (3) and (4) we see:

\[
\text{Slope formula for the Tangent at } P
\]

\[
\lim_{Q \to P} \text{(slope formula of Secants PQ)}
\]

The **BIG LEAP** is to put in a third equal sign:

\[
\text{Slope formula for the Tangent at } P \quad \text{equals} \quad 2x
\]

What we have shown is worth emphasizing:

\[
\begin{array}{l}
\text{The formula for the slope} \\
\text{of the tangent at any point } P(x, y) \\
\text{on the curve } y = x^2
\end{array} = 2x
\]
J. USING THIS EASY FORMULA

Going back to our original problem: What is the slope of the tangent at the point P(1, 1) on the curve \( y = f(x) = x^2 \)?

Now we have the formula for the slope of the tangent. The formula says \( \text{slope} = 2x \)

At point P, we have \( x=1 \), so

Slope of Tangent at P(1, 1) = 2 \times 1 = 2

This is the answer we predicted in Section F! 😊

The beauty of a formula is that it is now just as easy to calculate the slope at (2, 4) or at (1.5, 2.25), or at (0, 0), or at any point on the curve that we choose.

At the point (-2, 4), \( x = -2 \)

So, slope of tangent there

\[
\text{slope} = 2x \\
= 2 \times -2 \\
= -4
\]

At the point (3, 9), \( x = 3 \)

So, slope of tangent there

\[
\text{slope} = 2x \\
= 2 \times 3 \\
= 6
\]
The different notations for
"the formula for the slope of the tangent at the point (x, y) on the curve \( y = f(x) \)" are

\[
\frac{dy}{dx} \quad \text{or} \quad f'(x) \quad \text{or} \quad y' \]

We pronounce them “Dee y Dee x” “f prime of x” “y prime of x”

and this slope formula is called the (first) derivative.

We can write the following:

If \( y = f(x) = x^2 \), then

\[
\frac{dy}{dx} = 2x \quad \text{or} \quad f(x) = 2x \quad \text{or} \quad y' = 2x
\]

[Notice that the derivative of a function of \( x \) is again a (another) function of \( x \).]
K. OTHER FUNCTIONS, OTHER DERIVATIVES

Every calculus book contains shortcuts (called theorems, or rules) for finding the derivative of a function. Several of the most important ones are listed below. These rules depend upon the form of the function.

| Rule (1) | If \( y = f(x) = c \), \( c \) being a constant, then \( f'(x) = 0 \).
| “The derivative of a constant equals zero.” |
| Rule (2) | If \( y = f(x) = cx^n \), \( c \) and \( n \) being constants, then \( f'(x) = c \cdot n \cdot x^{n-1} \)
| “The derivative of a constant times \( x \) to the power \( n \) equals the constant, \( n \), times \( x \) to the one less power.” |
| Rule (3) | If \( y = f(x) \pm g(x) \), a sum or difference of 2 functions, then \( \frac{dy}{dx} = f'(x) \pm g'(x) \)
| “We differentiate term-by-term.” |

Examples

**Differentiating with the Rules**

- Rule (2) applies in finding the derivative for \( y = x^2 \). In this case, \( c=1 \) and \( n=2 \)

Therefore, \( \frac{dy}{dx} = 1 \cdot 2 \cdot x^{2-1} = 2x = 2x \)

- We need to use all three rules to determine the derivative for a linear function.

Consider the linear cost function \( TC = 10q + 500 \)

\[
TC' = (\text{the derivative of } 10q) + (\text{the derivative of } 500) \\
= (10 \cdot 1 \cdot q^{1-1}) + (0) \\
= 10
\]

Since \( q^0 = 1 \)

We observe that for a linear function, the derivative is equal to the slope. That is, For the function \( y = mx + b \), the derivative \( y' = m \)
It is not necessary for you to memorize these Rules, or even to spend time practicing them. They are summarized all over; check the index of any book containing calculus, check the Calculus section of the “Quantitative Skills Interactive” CD.

THE IMPORTANT NOTION HERE IS THAT THE DERIVATIVE OF A FUNCTION IS JUST THE FORMULA FOR THE SLOPE OF THE TANGENT TO THAT CURVE AT ANY POINT.

WE SOMETIMES ABBREVIATE THIS NOTION BY SAYING THAT THE DERIVATIVE IS THE SLOPE AT A POINT.

Slope at P = derivative evaluated at P
L. THE DERIVATIVE = THE INSTANTANEOUS RATE

As we discussed in Sections A and B, the slope of a line may be interpreted as the rate of change of one variable with respect to another. Rate is constant in this case.

How should we interpret the ever changing tangent slopes for a curve? We will interpret the slope of the tangent at a point P as instantaneous rate of change of y with respect to x. The slope for a curve gives a quick snapshot showing what is happening at point P.

Remember that the derivative is the (formula for the) slope of the tangent at a point. So the derivative is also a formula for rate.

Let's look at a graph: Note that the curve is decreasing through P. What is the sign of the derivative at point P? Draw the Tangent to the curve at P. Observe that the slope of the Tangent at P is negative. So the derivative would be negative at point P, too.

With a linear function, when the slope is negative, y is decreasing as x increases. So it is with a curve: a negative derivative (slope) at a point means the function is decreasing around that point. And a positive derivative (slope) at a point means the function is increasing around that point.
Example  Here we will use the function $y = x^2$ as a model for sales of widgets.

Suppose $x$ represents years since 2000 and $y$ represents thousands of widgets sold.

For instance, in 2001, $x=1$. Since $y = f(1) = 1^2 = 1$, we know 1,000 widgets were sold in 2001.

How fast were sales changing in 2001?

The instantaneous rate of change of sales in 2001 ($x=1$) is given by the slope of the tangent at $x=1$ which we showed (Section J) was +2.

Thus, sales were changing (increasing, since the slope is positive) at a rate of 2,000 widgets per year in 2001.
How were sales changing in 1998 ($x = -2$)?

Since $f''(x) = 2x$, it follows that $f''(-2) = 2 \cdot -2 = -4$. This means that sales were decreasing at the rate of 4,000 widgets per year in 1998.

To summarize:

1. Slope is positive when the curve is rising. When the curve is rising, the slope is positive.
2. Slope is negative when the curve is falling. When the curve is falling, the slope is negative.
M. ‘MARGINAL’ = RATE

In business and economics, the word **marginal** is used to refer to the rate of change. We would say that **marginal sales** of widgets were 2,000 per year in 2001.

**Example:** A manufacturer is not only interested in the Total Cost, \( TC(q) \), at a certain production level \( q \), but she is also interested in the **Marginal Cost**, \( MC(q) \). \( MC(q) \) is the rate of change of cost at production level \( q \). \( MC(q) \) tells how fast cost is increasing at \( q \).

For instance, if \( MC(100) = $15 \), we can say that when 100 widgets have been produced, Total Cost is increasing at the rate of $15 per additional widget produced.

*This means that the (approximate) incremental cost due to producing an additional widget is $15.*

**Graphic Example** Look at the graph of a non-linear cost function below. Notice that Total Cost is increasing faster at point \( A \) than at \( B \). In fact, Total Cost is increasing at a **decreasing** rate up until point \( P \). Another way to say this is **Marginal Cost is decreasing** up to the production level corresponding to point \( P \).

![Graph of total cost function](image)

Notice that Total Cost is increasing faster at point \( N \) than at \( M \). After the production level corresponding to point \( P \), Total Cost is increasing at an **increasing** rate. So **Marginal Cost is increasing** from this production level on.
N. USING SLOPE FINDING TO DETERMINE MAXIMUM AND MINIMUM VALUES --
OPTIMIZATION THEORY AND CURVE SKETCHING

Now that we can find the slope at any point on a curve we will concentrate on points whose tangent slope is zero. If the tangent at a point P has zero slope, the sketch of the curve has to look like one of these four below:

These points are called Critical Points.

Important Question: How do we find minimum and maximum points for a general curve \( y = f(x) \)?

We look for those points with horizontal tangents. (i.e., Critical Points). Horizontal tangents have zero slope. Since the slope of the tangent at a point equals the derivative evaluated there, we look for those points at which the derivative \( f'(x) \), is zero.

1. Find \( f' \) [As you recall, the first derivative can also be written as \( \frac{dy}{dx} \) or \( y' \)]

2. Determine all the critical values, that is to say, all the values of \( x \) where \( f' = 0 \)

3. Once we find points of zero slope, we will test for one of these cases:
Example 1: Let's re-examine the function \( f(x) = x^2 \). Suppose we do NOT know what the graph looks like.

1. The first derivative is \( f'(x) = 2x \).

2. Set the first derivative equal to zero and solve for \( x \):
   \[
   2x = 0 \\
   \text{So } x = 0
   \]
   Therefore, at the point where \( x = 0 \), [i.e. at (0, 0) ] we know that the curve has a minimum, a maximum, or an inflection point. Which is it?

3. Notice that:
   - To the left of \( x = 0 \), \( x \)-values are negative. So \( f'(x) = 2x \) is negative. Therefore tangents slope downward.
   - To the right of \( x = 0 \), \( x \)-values are positive. So \( f'(x) = 2x \) is positive. Therefore tangents slope upward.

Therefore, we can sketch the graph of \( y = x^2 \):

![Graph of y = x^2](image)

Question: Can there be any other minimum or maximum points on this graph?

NO, because \( f'(x) \) is zero only for \( x = 0 \). So the graph must look like we have drawn it!
Example 2: Next, we will examine the function $y = f(x) = x^3 - 6x^2 + 9x$

We do NOT know what the curve looks like in this case.

(1) The first derivative is $f'(x) = 3x^2 - 12x + 9$.

(2) Set $f'(x)$ equal to zero and solve for $x$:

$$3x^2 - 12x + 9 = 0$$

$$x^2 - 4x + 3 = 0$$

$$(x-3)(x-1) = 0$$

So we find that in this particular case there are two points at which the slope is zero, at $x = 1$ and at $x = 3$.

The two points are: $(1, f(1)) = (1 , 1^3 - 6\cdot1^2 + 9\cdot1) = (1 , 4)$

and

$$(3, f(3)) = (3 , 3^3 - 6\cdot3^2 + 9\cdot3) = (3 , 0 )$$

(3) Notice that:

- To the left of $x = 1$, $f'(x) = (x-3)(x-1)$ is positive, because $(x-3) \ast (x-1) = Positive$. Therefore tangents slope upward.

- To the right of $x = 1$, $f'(x) = (x-3)(x-1)$ is negative, because $(x-3) \ast (x-1) = Negative$. Therefore tangents slope downward.

- So the graph at $(1, 4)$ must look like this:

- To the left of $x = 3$, $f'(x) = (x-3)(x-1)$ is negative, because $(x-3) \ast (x-1) = Negative$. Therefore tangents slope downward.

- To the right of $x = 3$, $f'(x) = (x-3)(x-1)$ is positive, because $(x-3) \ast (x-1) = Positive$. Therefore tangents slope upward.

- So the graph at $(3,0)$ must look like:
Let’s sketch the curve \( y = x^3 - 6x^2 + 9x \)

Notice that for this function there are two points of zero slope. They are called *relative maximum and relative minimum points* since the curve does extend lower and higher in value than at these two points.

(A second approach for determining whether a point is a minimum or a maximum point is by plotting points on either side of the point in question. There is also a third approach, The Second Derivative Test. This essay does not cover this test.)
O. A BUSINESS APPLICATION

Revenue Maximization Suppose a manufacturer finds that the Revenue she receives from selling her product depends upon the unit price of the product. That relationship between Revenue \( R \) and unit price \( p \) is

\[
R = f(p) = 200p - 6.7p^2
\]

Notice that if \( p \) is set low at $5, Revenue will equal:

\[
R = 200(5) - 6.7(5)^2 \\
= 1000 - 167.5 \\
= $832.50
\]

But if the price is set higher at, say, $27:

\[
R = 200(27) - 6.7(27)^2 \\
= 5400 - 4884.30 \\
= $515.70
\]

Clearly, $27 is too high a price. Should \( p \) be set at $5? What is the price that gives the largest Revenue?

So we know that we are trying to locate a maximum point on the curve.

That maximum point has a horizontal tangent. That means we want to determine the point with zero slope. And that means we want the point where the derivative is zero.

The derivative takes the form:

\[
(1) \frac{dR}{dp} = 200 - 13.4p
\]

(2) Then we set \( \frac{dR}{dp} = 0 \) and solve the equation for \( p \).

\[
200 - 13.4p = 0
\]

\[
200 = 13.4p
\]

so \( p = $14.93 \)
What is the Revenue obtained at the revenue-maximizing price of \$14.93? 

We calculate:

\[ R = 200 (14.93) - 6.7 (14.93)^2 \]
\[ = \$1492.54 \]

Notice that:

- To the left of \( p = \$14.93 \), \( \frac{dR}{dp} = 200 - 13.4p \) is positive; therefore tangents slope upward.
- To the right of \( p = \$14.93 \), \( \frac{dR}{dp} = 200 - 13.4p \) is negative. Therefore tangents slope downward.

So the graph must look like:

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So Revenue is *increasing* for prices less than \$14.93. At \( p = \$5 \), for example, an increase in price will cause \( R \) to increase.

But for prices greater than \$14.93, for example at \( p = \$27 \), an *increase* in price causes a *decrease* in Revenue.
This Revenue Maximization problem introduces you to the use of calculus in what are termed Optimization Problems. Calculus can help answer the following types of questions:

- Where should a new hospital be located in order to service a particular area if travel time to the hospital is to be minimized?

- How should a large pharmaceutical firm allocate its expenditures for TV, Internet and magazine advertising?

- What quantity of an item should be ordered, and how often, so that inventory and ordering costs are kept to a minimum?

- How long should an airline expect to keep its Boeing 707’s airplanes?

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Note: In order to apply calculus techniques, however, we must be able to represent the real world problem by mathematical functions—sometimes this is very hard to do!! But in any case, the ideas of rate, marginal, and optimization are at the very foundation of business decision making.
P. CONGRATULATIONS!!

YOU NOW HAVE THE MAIN IDEAS OF DIFFERENTIAL CALCULUS!!

1) For a line, the rate of change is constant. The rate of change is given by the slope.

2) For a non-linear curve, the rate of change is continually changing. We define the *instantaneous rate of change at a point* as the slope of the tangent to the curve at the point.

3) The slope of the tangent to the curve at a point is given by the derivative evaluated at that point.

4) The following three concepts are equivalent for a curve \( y = f(x) \)
   - The slope of the tangent to the curve at a point
   - The derivative of the function evaluated at that point
   - The instantaneous rate of change at that point

5) A maximum or minimum point may be found by setting the derivative equal to zero. Solving for \( x \) will then give the point(s) with horizontal tangents [Critical Point(s)]. To determine whether a critical point is a minimum or a maximum point, check the slopes of the tangents on either side.

6) The term *Marginal Cost* refers to the derivative of the Total Cost. In economics and business, the terms *marginal cost, marginal revenue, marginal profit, …* refers to the to the *incremental cost, incremental revenue, incremental profit, …* derived from the next unit, and thus *marginal* gives a rate.